Math 246C Lecture 15 Notes

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1 Existence of a Dipole Green's Function

1.1 Symmetry of Green's functions

Proposition 1.1 (symmetry of Green's functions)**.** *Let X be a Riemann surface such that G_x exists for some $x \in X$. Then G_y exists for any y, and $G_x(y) = G_y(x)$.*

We have already proven this when X is simply connected.

Proof. Idea: Let $\tilde{X}$ be a universal covering space of X. On $\tilde{X}$, $G_{\tilde{z}}$ exists for all $\tilde{z} \in p^{-1}(x)$, where $p : \tilde{X} \to X$ is a covering map. So $\tilde{X} = D$, and

$$G_{\tilde{z}}(\tilde{y}) = \log \left| \frac{1 - \overline{\tilde{z}}\tilde{y}}{\tilde{y} - \tilde{z}} \right|$$

is symmetric. $\square$

Remark 1.1. It follows that any Riemann surface is second countable (Rado's theorem). Take X, and remove a parametric disc. Then the rest of the space has a Green's function, so it is covered by a disc, which is second countable.

1.2 Existence of a dipole Green's function

Theorem 1.1 (existence of a dipole Green's function)**.** *Let X be a Riemann surface, and let $x_1 \neq x_2 \in X$. Let $z_j : D_j \to \{|z| < 1\}$ be parametric discs such that $z_j(x_j) = 0$ and $\overline{D}_1 \cap \overline{D}_2 = \varnothing$. Then there exists a harmonic G_{x_1,x_2} on $X \setminus \{x_1, x_2\}$ such that $G_{x_1,x_2} + \log |z_1(y)|$ is harmonic in D_1, $G_{x_1,x_2} + \log |z_2(y)|$ is harmonic in D_2, and $\sup_{X \setminus (D_1 \cup D_2)} |G_{x_1,x_2}| < \infty$.*

Proof. Let $D_0 \subseteq X$ be a parametric disc $z_0 : D_0 : \{|z| < 1\}$ with $z_0(x_0) = 0$ and $\overline{D}_0 \cap \overline{D}_j = \varnothing$ for $j = 1, 2$. For $0 < t < 1$, let $tD_0 = \{y \in D_0 : |z_0(y)| < t\}$. Let $X_t = X \setminus \overline{tD_0}$. We know that Green's function $G_{X_t}(x_1, y)$ exists for all $y \in X_t \setminus \{x_1\}$ and for all t. Let $0 < r < 1$. Let $v \in \mathcal{F}_{x_1}$, the Perron family on X_t used to construct $G_{X_t}(x_1, y)$. When $y \in X_t \setminus \overline{rD_1}$,

$$v(y) \leq \sup_{\partial(rD_1)} v$$

by the maximum principle. Taking the sup over all $v \in \mathcal{F}_{x_1}$,

$$G_{X_y}(x_1, y) \le \sup_{\partial(rD_1)} G_{X_t}(x_1, y) =: M(t).$$

On the other hand, we have shown last time that

$$\sup_{\partial(rD_1)} v + \log(r) \le \sup_{\partial D_1} v$$

(by applying the maximum principle to $v(y) + \log|z_1(y)|$ in D_1). We get

$$M(t) + \log(r) \le \sup_{\partial D_1} G_{X_t}(x_1, y).$$

Consider the function

$$u_t(y) = M(t) - G_{X_t}(x_1, y), \qquad y \in X_t \setminus \overline{rD_1}$$

Then $u_t(y) \ge 0$ and is harmonic. There exists a $y_0 \in \partial D_1$ such that $u_t(y_0) \le \log(1/r)$. We want to apply Harnack's principle to u_t: Let $K \subseteq X_1 \setminus \overline{rD_1}$ be compact such that $\overline{D}_2 \subseteq K_1$ and $\partial D_1 \subseteq K$. By Harnack's inequality,

$$\frac{\sup_K u_t}{\inf_K u_t} \le C(K, r),$$

where $C(K, r)$ is a geometric constant independent of t. So

$$u_t(y) \le C, \qquad y \in K,$$

uniformly in t. So

$$|G_{X_t}(x_1, y) - G_{X_t}(x_1, x_2)| = |u_t(y) - u_t(x_2)| \le 2C.$$

Similarly,

$$|G_{X_t}(x_2, y) - G_{X_t}(x_2, x_1)| \le 2C, \qquad y \in K', K' \supseteq \overline{D}_1 \cup \partial D_2.$$

By the symmetry of Green's functions, $G_{X_t}(x_2, x_1) = G_{X_t}(x_1, x_2)$. So we get

$$|G_{X_t}(x_1, y) - G_{X_t}(x_2, y)| \le C$$

uniformly in t for $y \in \partial D_1 \cup \partial D_2$.

We also want uniform control on G_t on $X_t \setminus (D_1 \cup D_2)$: Let $v \in \mathcal{F}_{x_1}$. Then $v(y) - G_{X_t}(x_2, y)$ is subharmonic for $y \in X_t \setminus \overline{D}_1$, so

$$v(y) - G_{X_t}(x_2, y) \le \sup_{\partial D_1}(v - G_{X_t}(x_2, y)) \le C$$

by the maximum principle. So

$$\underbrace{G_{X_t}(x_1,y) - G_{X_t}(x_2,y)}_{:=G_t(y,x_1,x_2)} \leq C$$

on $X_t \setminus D_1$. Similarly,

$$\inf_{y\in X_t\setminus D_2} G_t = -\sup_{X_t\setminus D_2} -G_t \geq C,$$

so we get

$$\sup_{X_t\setminus(D_1\cup D_2)} |G_t| \leq C,$$

uniformly in t. In D_j, $j=1,2$, $G_t(y,x_1,x_2)+\log|z_1(y)|$ is harmonic in D_1. By the maximum principle applied in D_1,

$$|G_t(y,x_1.x_2)+\log|z_1(y)|| \leq C, \qquad y\in D_1,$$

uniformly in t. Similarly,

$$|G_t(y,x_1,x_2)-\log|z_2(y)|| \leq C, \qquad y\in D_2,$$

uniformly in t.

These three uniform inequalities give us the following: Let $K\subseteq X\setminus\{x_1,x_2,x_0\}$ be compact. By normal families and Rado's theorem, there exists a sequence $t_n\to 0$ and G harmonic on $X\setminus\{x_0,x_1,x_2\}$ such that $G_{t_n}\to G$ locally uniformly on $X\setminus\{x_0,x_1,x_2\}$. The first inequality gives us that G is bounded in $D_0\setminus\{x_0\}$; so G extends harmonically to D_0. Similarly,

$$|G(y)+\log|z_1(y)|\leq C \text{ in } D_1 \implies G+\log|z_1| \text{ is harmonic in } D_1,$$

$$|G(y)+\log|z_2(y)|\leq C \text{ in } D_2 \implies G+\log|z_2| \text{ is harmonic in } D_2.$$

So G is a dipole Green's function. $\square$